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Supersonic laminar boundary layer near the plane of symmetry of a cone at incidence

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Supersonic laminar boundary-layer equations near the plane of symmetry of a cone at incidence are treated by the similarity method. Numerical integration of differential equations governing such a flow is performed, taking into consideration the temperature dependence of the Prandtl number *Pr* and viscosity μ throughout the boundary layer. On the leeward side, a detailed consideration of the solutions shows the existence of two solutions up to a critical incidence beyond which it appears that no solution may be found. Calculations carried out for a set of values of the external flow Mach number show up a significant effect of this parameter on the behaviour of the boundary layer.

1. Introduction

The existence of similar solutions for the supersonic laminar boundary layer near the symmetry plane of a circular cone at an angle of attack has been demonstrated by Hayes (1951) and Moore (1953). Such solutions are used by many authors including Reshotko (1957), Cooke (1966), Vvedenskaya (1966), Boericke (1970) and Roux (1971). The difficulties encountered in determining these solutions on the leeward side are particularly emphasized by Moore (1953). By taking the Prandtl number equal to 1, disregarding heat transfer and making an asymptotic solution analysis ($\eta \rightarrow \infty$), he pointed out an azimuthal velocity gradient effect upon the existence and uniqueness of solutions. In the present paper, the numerical integration of the system of differential equations governing the similar boundary-layer solutions, carried out for several values of a parameter M', shows the existence of two solutions for M' between 0 and a critical value M'_{cr}; the parameter M', as defined by equation (9) below, is proportional to the azimuthal velocity gradient. It appears that no solution can be found for M' less than M'_{cr} and that for $M' = M'_{cr}$ there is only one solution for the system. A detailed numerical integration of the solutions is performed for several Mach numbers and different I_p/I_{0e} values, where I is the static enthalpy and the subscripts p, 0 and e refer to the wall condition, stagnation condition and external flow respectively. Thus it is possible to state precisely the effect of these two parameters on the value of M'_{cr}. The variation of M'_{cr} with M_e is obtained for $I_p/I_{0e} = 0{\cdot}5$. We are carrying out a similar investigation for other values of this parameter.

These results differ from the criteria for existence and uniqueness of solutions proposed by Moore (1953), thus confirming the criticisms formulated by Boericke (1970).

On the other hand, as far as the solutions in the hypersonic limiting case, as obtained by Trella & Libby (1965), are concerned it must be stressed that their range of application when $M_e \gg 1$ is limited to the case of slender cones and that in this case solutions on the leeward side only exist for very small incidence, this incidence becoming smaller as M_e increases.

In the windward plane of symmetry ($M' > 0$), the similar solutions are useful as initial solutions for the determination of the boundary layer on both sides of the symmetry plane, by finite-difference methods for example.

2. Governing equations

The governing equations of motion are taken with the usual boundary-layer assumptions; their expression for a cone has been given by the author (1971) following a previous report by Cooke (1966). Moreover it is assumed that: (i) the flowing gas behaves as a perfect gas, (ii) the external flow is conical and isentropic, (iii) the wall is isothermal, (iv) the viscosity coefficient μ and thermal conductivity k are dependent on temperature following the Sutherland law and a semi-empirical law given in the NBS Tables (Tables of thermodynamic and transport properties, NBS 1960) respectively.

The existence of similar solutions for the resulting system of equations has been shown by Hayes and Moore. In the symmetry plane we are seeking the following form for such solutions:

$$U = U_e u(\eta), \tag{1}$$

$$(dW/d\phi) = (dW_e/d\phi)\, w(\eta), \tag{2}$$

$$I = I_e T(\eta), \tag{3}$$

where η is the similarity variable defined by

$$\eta = \left(\frac{U_e}{\rho_e \mu_e}\right)^{\frac{1}{2}} x^{-\frac{1}{2}} \int_0^y \rho\, dy \tag{4}$$

and U is the meridional component of velocity, W the circumferential velocity, u, w transformed components of velocity and T the static enthalpy ratio, I/I_e. The system reduces to a set of ordinary differential equations.

$$dv/d\eta = -\tfrac{3}{2}u - wM', \tag{5}$$

$$\frac{d}{d\eta}\left(C\frac{du}{d\eta}\right) = v\frac{du}{d\eta}, \tag{6}$$

$$\frac{d}{d\eta}\left(C\frac{dw}{d\eta}\right) = v\frac{dw}{d\eta} + w(u + M'w) - (1+M')T, \tag{7}$$

$$\frac{d}{d\eta}\left(\frac{C}{Pr}\frac{dT}{d\eta}\right) = v\frac{dT}{d\eta} - C(\gamma-1)\,M_e^2\left(\frac{du}{d\eta}\right)^2, \tag{8}$$

where $C = \rho\mu/\rho_e\mu_e$ and the parameter M' is defined by

$$M' = \frac{1}{U_e \sin\theta_c}\frac{dW_e}{d\phi}, \tag{9}$$

where θ_c is the semi-vertex angle of the cone. The boundary conditions are

$$u = v = w = 0, \quad T = T_p = I_p/I_e, \quad \text{for} \quad \eta = 0, \tag{10}$$

$$u \to 1, \quad w \to 1, \quad T \to 1, \quad \text{as} \quad \eta \to \infty. \tag{11}$$

The above system forms a set of seven differential equations of the first order for the following variables: u, $C\,du/d\eta$, w, $C\,dw/d\eta$, v, T, $(C/Pr)\,dT/d\eta$ (where C and Pr are explicit functions of temperature and hence of T and I_e). In order to use standard numerical methods for such a system the two-point boundary conditions are transformed into initial ones. It is therefore necessary to use an iterative process to correct all assumed initial conditions (that is, values of $C\,du/d\eta$, $C\,dw/d\eta$, $(C/Pr)\,dT/d\eta$ at $\eta = 0$) until the conditions for $\eta \to \infty$ are fulfilled. The iterative process chosen is similar to the one proposed by Dewey & Gross (1967) and has already been employed in the previous note Roux & Rey (1970).

It should be mentioned that when the temperature dependence of C and Pr is taken into account the system (5)–(11) not only contains the parameters M_e and M', which appear explicitly, but also a third parameter I_e. I_e is determined for given values of $T_p = I_p/I_e = (I_p/I_{0e})\,(1+\frac{1}{2}(\gamma-1)M_e^2)$ and I_p. The present calculations are carried out with $I_p/I_{0e} = 0{\cdot}5$ and with a value of the enthalpy I_p corresponding to a wall temperature of 300 °K.

Remarks concerning numerical calculation of the solutions

As we shall see below, the calculations bring out two solutions for any value of M' such that $M'_{cr} < M' \leqslant 0$. The solutions of 'type 1', permitting continuity at $M' = 0$ with the solutions corresponding to $M' > 0$, are obtained after 3 or 4 iterations; the running time is then about 5 min when $0 \leqslant \eta \leqslant 6$ and $\Delta\eta = 0{\cdot}1$. With Pr and C kept constant, this time is half as long. The determination of solutions of 'type 2' is much more difficult and one needs to guess the assumed initial conditions to within a certain accuracy; this difficulty is avoided by performing calculations gradually, from solutions of type 1. Furthermore, it is necessary to make the integration interval larger as $M' \to 0$ ($0 \leqslant \eta \leqslant 10$ when $|M'| < 0{\cdot}01$). At the same time the step size $\Delta\eta$ has to be smaller to keep the precision constant. To obtain the solutions, 6 or 7 iterations are then needed and the running time may reach 20 min when working with double precision on IBM 360–44.

3. Existence of two classes of solutions

For each numerical value of the Mach number considered, integration of the system (5)–(8) is first carried out for $M' > 0$. Type 1 solutions are then sought for M' varying gradually from 0 to a negative critical value M'_{cr} beyond which the numerical process is no longer convergent. The results are shown on table 1.

Type 2 solutions are afterwards computed in the neighbourhood of M'_{cr} and sought for M' gradually increasing up to a value very near 0. A detailed investigation performed for Mach number 5, up to $M' = -10^{-4}$, shows (table 2) that u'_p and T'_p tend to a finite positive limit and that w'_p behaves as $|M'|^{-a}$ with $a \to 1$. In the limiting case, $M' = 0$, the routine used previously for the equations (5)–(8) does not give a solution of type 2, since $w'_p \to \infty$. However, the behaviour of w'_p when $M' \to 0$ found previously, led us to believe that for $M' = 0$ the exponent a is equal to 1, and thus that a solution exists for which the product wM' is not zero. The existence of such a solution is confirmed by a particular study described below.

M'	u'_p	w'_p	T'_p
1·0000	1·00211	4·2124	1·9916
0·5000	0·84101	3·7514	1·6652
0·0000	0·61422	3·3638	1·2029
−0·1000	0·53634	3·4129	1·0427
−0·1500	0·46650	3·6253	0·8984
−0·1600	0·43478	3·7950	0·8326
−0·1620	0·42115	3·8848	0·8044
−0·1625	0·41453	3·9323	0·7906

TABLE 1. Type 1 solutions with $M_e = 5$, $I_p/C_p = 300$ °K, $I_{0e}/C_p = 600$ °K

M'	u'_p	$M'w'_p$	T'_p
−0·1600	0·37702	−0·68090	0·71294
−0·1500	0·34010	−0·70140	0·63675
−0·1250	0·28436	−0·65923	0·52293
−0·1000	0·23961	−0·65936	0·43333
−0·0500	0·15431	−0·51810	0·13702
−0·0100	0·07304	−0·29256	0·12916
−0·0010	0·04707	−0·19751	0·08764
−0·0001	0·04388	−0·18477	0·08261
0·0000	0·04351	−0·18329	0·08204

TABLE 2. Type 2 solutions with $M_e = 5$, $I_p/C_p = 300$ °K, $I_{0e}/C_p = 600$ °K

Two classes of solutions are hence found for $M'_{cr} < M' \leqslant 0$. This result is in disagreement with the conclusion concerning the existence and the uniqueness of the solutions obtained by Moore (1953) in his analytical study of the asymptotic behaviour (for $\eta \to \infty$) of the solutions. Some criticisms have been formulated by Boericke (1970) about the validity of the criteria suggested by this author when assuming $Pr = 1$ and disregarding heat transfer. In our case, when extending the Moore's analysis to equations (5)–(8), we find nearly the same expressions for w as those obtained by this author (Moore 1953, equations 26(a), 26(b)) and we could also obtain similar criteria which would not be consistent with the numerical results. Consequently, it appears that the determination of the asymptotic behaviour of the solutions as investigated by many authors after Hartree (1937) does not make them able to prove the existence and uniqueness of the solutions directly. On the contrary, this investigation must only be used to obtain an analytical expression, when $\eta \to \infty$, for a solution already known (by a numerical method for example) up to some sufficiently large value of η for which the boundary conditions are nearly fufilled.

Physical foundation of the solutions

Of the two classes of solutions found here, type 2 solutions do not have any physical foundation. To demonstrate this it seems to be sufficient to show that these solutions do not tend to the zero incidence solutions when $M' \to 0$.

When the cone incidence is zero M' is equal to 0. A similar analysis, which has been carried out by the author (1971), gives the following equations for the boundary layer in this case.

$$dv/d\eta = -\tfrac{3}{2}u, \tag{12}$$

$$\frac{d}{d\eta}\left(C\frac{du}{d\eta}\right) = v\frac{du}{d\eta}, \tag{13}$$

$$\frac{d}{d\eta}\left(\frac{C}{Pr}\frac{dT}{d\eta}\right) = v\frac{dT}{d\eta} - C(\gamma-1)M_e^2\left(\frac{du}{d\eta}\right)^2, \tag{14}$$

with the boundary conditions

$$u = v = 0, \quad T = I_p/I_{0e}, \quad \text{when} \quad \eta = 0, \tag{15}$$

$$u \to 1, \quad T \to 1, \quad \text{as} \quad \eta \to \infty. \tag{16}$$

If the Mach number M_e is given this system admits only one solution.

If we consider the type 1 solutions for the limiting case $M' = 0$, equations (5), (6) and (8) become the same as (12), (13) and (14) since $w(\eta)$ keeps a finite value when $M' \to 0$. Equation (7) then takes the following form:

$$\frac{d}{d\eta}\left(C\frac{dw}{d\eta}\right) - v\frac{dw}{d\eta} - wu = T, \tag{17}$$

where the variables $u(\eta)$, $v(\eta)$, $T(\eta)$ are independent of $w(\eta)$ and regular on the whole integration interval $0 \leqslant \eta \leqslant \eta_{\max}$. Therefore $w(\eta)$, which is a solution of the linear differential equation (4), has the property of being regular on the whole integration interval (Whittaker & Watson 1943, §§10–21). The $w(\eta)$ values are then finite and consequently $dW/d\phi$ is equal to zero because $dW_e/d\phi = 0$ when the angle of attack i is zero. The solutions of (5)–(8) for $M' = 0$ are indeed available in the zero incidence case.

With the type 2 solutions, on the other hand, when $M' \to 0$ the product $M'w(\eta)$ has been found to be of the same order as $(M')^{1-a}$ and thus does not tend to zero when the exponent a is 1. In this case, the system (5)–(8) must be written as follows after multiplying all terms of (7) by M' and setting $\overline{w} = wM'$.

$$dv/d\eta = -\tfrac{3}{2}u - \overline{w}, \tag{18}$$

$$\frac{d}{d\eta}\left(C\frac{du}{d\eta}\right) = v\frac{du}{d\eta}, \tag{19}$$

$$\frac{d}{d\eta}\left(C\frac{d\overline{w}}{d\eta}\right) = v\frac{d\overline{w}}{d\eta} + \overline{w}(u+\overline{w}) - (1+M')M'T, \tag{20}$$

$$\frac{d}{d\eta}\left(\frac{C}{Pr}\frac{dT}{d\eta}\right) = v\frac{dT}{d\eta} - C(\gamma-1)M_e^2\left(\frac{du}{d\eta}\right)^2, \tag{21}$$

with the boundary conditions

$$u = v = \overline{w} = 0, \quad T_p = I_p/I_{0e}, \quad \text{when} \quad \eta = 0, \tag{22}$$

$$u \to 1, \quad \overline{w} \to 0, \quad T \to 1, \quad \text{as} \quad \eta \to \infty. \tag{23}$$

This system is integrated by a numerical method quite similar to the previous one. When $M' = 0$ the system (16)–(19) admits a solution $\overline{w}$ different from zero which is calculated in both cases of variable and constant ($C = 1, Pr = 1$) transport properties, for all the Mach numbers M_e previously considered. The values of the derivatives at the wall of the solutions computed for $M_e = 5$ are added at the bottom of the tables 2 and 4.

Thus, in the case of type 2 solutions the system (5)–(8) is not reducible, on letting $M' \to 0$, to the form (12)–(14), valid for the zero incidence. Moreover, the solutions performed when integrating (18)–(21) in the limiting case $M' = 0$, show that $wM' \neq 0$. Consequently $dW/d\phi$ is different from zero when $i = 0$, a result which has no physical meaning.

The type 2 solutions are not physically sound but their mathematical existence compels us to find out to which class a calculated solution belongs. For this purpose an additional investigation is necessary. We suggest, for example, performing a calculation for an M' value very near the one considered and looking at the increments ratio $\Delta u_p'/\Delta M'$. Indeed, according to numerical results obtained in this study this ratio is always positive for the type 1 solutions and negative in the other case, as we shall see below.

4. Mach number effect

The numerical values of the derivative at the wall for solutions obtained in the Mach number range $1 \leqslant M_e \leqslant 7$ are plotted against M' in figures 1–3. As an example, the numerical values of these variables computed for $M_e = 5$ are listed in tables 1 and 2. Computation of M'_{cr} is carried out with an accuracy of about $2{\cdot}5 \times 10^{-4}$. The results obtained for each Mach number make obvious a significant variation of M'_{cr} with M_e (see figure 4). We shall see below that when $M_e \to \infty$, $M'_{cr} \to 0$. As for the distribution of the variables throughout the boundary layer, the profiles of type 1 solutions are shown in figures 5–7 for several M' values. In the case of type 2 solutions only the U/U_e and I/I_e profiles are presented (see figures 8 and 9); the large variations of $(dW/d\phi)/(dW_e/d\phi)$ for the different M' values make it difficult to draw them on the same graph.

Influence of transport properties

As indicated in § 2, when the variation of the transport properties with temperature are taken into consideration the solutions depend on I_e and thence on I_p. The influence of I_p for $I_p/I_{oe} = 0{\cdot}5$ is studied by integration of (2)–(8) in the case

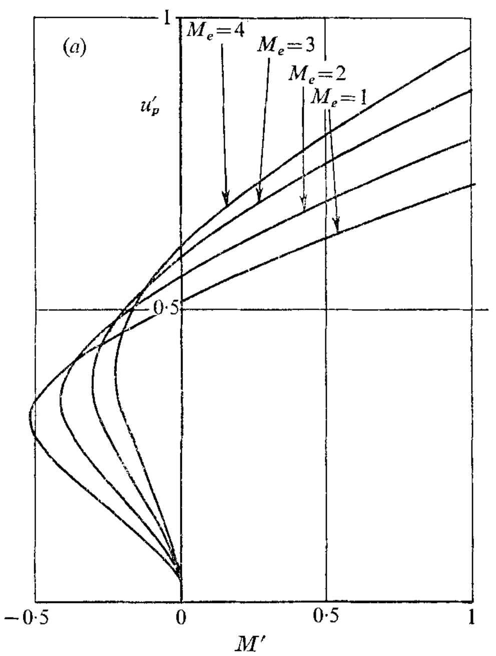

FIGURE 1. Streamwise wall-shear parameter *vs.* M' ($I_p/I_{0e} = 0{\cdot}5$).

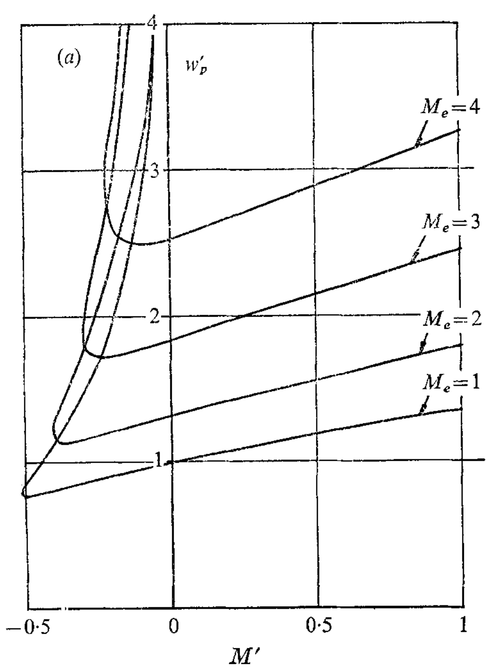

FIGURE 2. Cross-flow wall-shear parameter *vs.* M' ($I_p/I_{0e} = 0{\cdot}5$).

$M_e = 5$, taking I_p values corresponding to a wall temperature equal to 300 °K or to 500 °K. The difference between the M'_{cr} values obtained in both cases is less than 2 %. A set of calculations performed with $C = 1$ and $Pr = 1$, so that the solutions are independent of I_p, leads to an M'_{cr} value departing less than 6 %

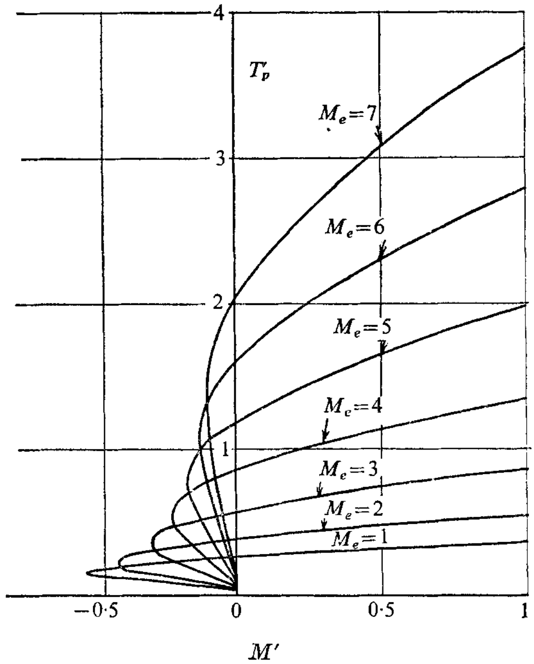

FIGURE 3. Heat-transfer parameter *vs.* M' ($I_p/I_{0e} = 0{\cdot}5$).

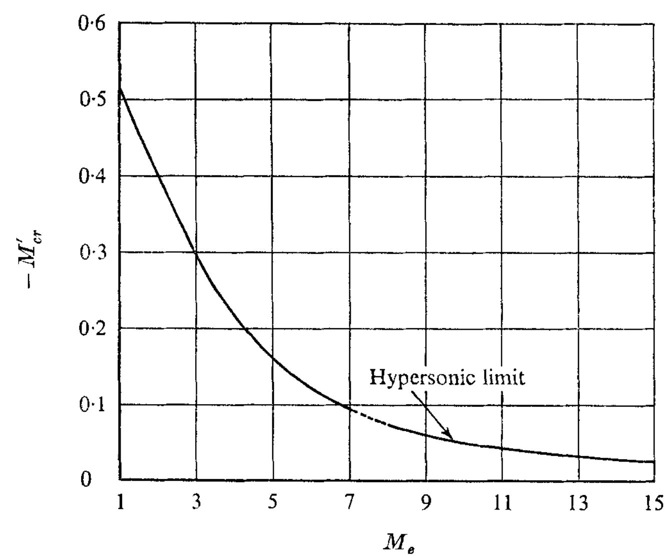

FIGURE 4. Effect of the external flow Mach number on the value of M'_{cr} ($I_p/I_{0e} = 0{\cdot}5$).

from those obtained in the two previous cases. The results performed for $M_e = 5$ are displayed in tables 3 and 4. Thus the M'_{cr} value depends essentially on the parameter M_e (and also on I_p/I_{0e} as indicated by the first results of the investigation which we are carrying on at present). The assumption of the transport properties variations does not give rise to a significant effect.

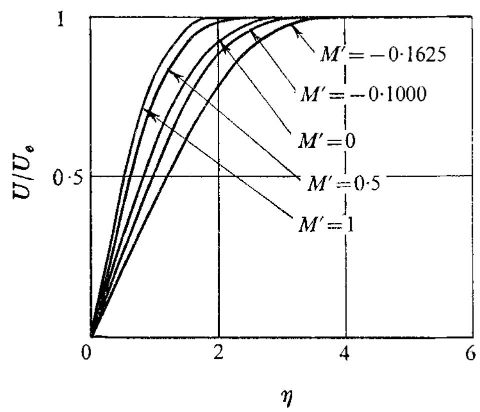

FIGURE 5. Streamwise velocity profiles, type 1 solutions.

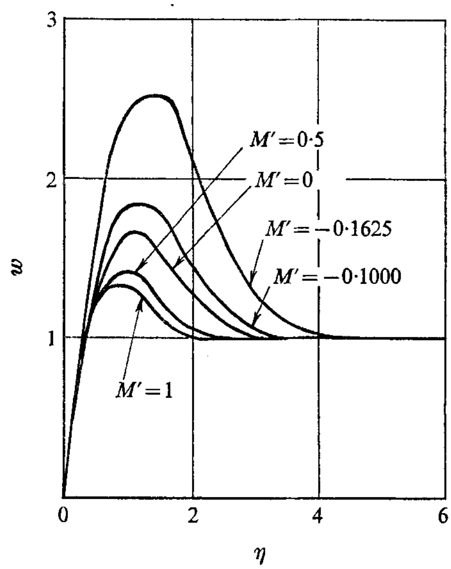

FIGURE 6. Cross-flow velocity profiles, type 1 solutions.

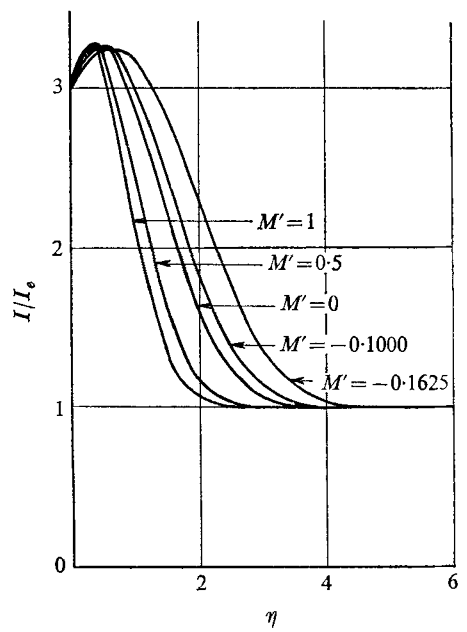

FIGURE 7. Temperature profiles, type 1 solutions.

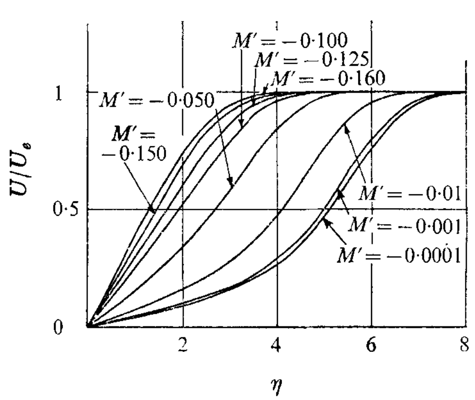

FIGURE 8. Streamwise velocity profiles, type 2 solutions.

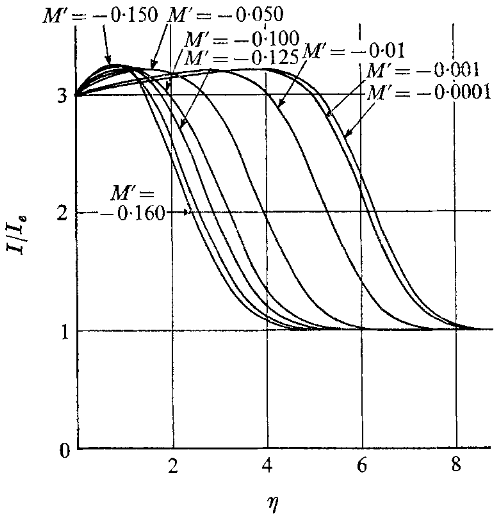

FIGURE 9. Temperature profiles, type 2 solutions.

M'	u'_p	w'_p	T'_p
1·0000	0·94449	4·0763	2·8335
0·5000	0·79181	3·6349	2·3754
0·0000	0·57514	3·2799	1·7254
−0·1000	0·49812	3·3557	1·4944
−0·1500	0·41545	3·6995	1·2464
−0·1520	0·40752	3·7646	1·2172
−0·1530	0·39846	3·8172	1·1954
−0·1535	0·39273	3·8612	1·1782

TABLE 3. Type 1 solutions with $M_e = 5$, $I_p/I_{0e} = 0{\cdot}5$, $C = 1$, $Pr = 1$

M'	u'_p	$M'w'_p$	T'_p
−0·1500	0·35034	−0·63933	1·05102
−0·1000	0·23793	−0·62499	0·71378
−0·0500	0·15271	−0·49429	0·45814
−0·0100	0·07088	−0·27745	0·21264
−0·0010	0·04464	−0·18494	0·13393
−0·0001	0·04143	−0·17249	0·12428
−0·00001	0·04110	−0·17119	0·12329
−0·000001	0·04106	−0·17106	0·12319
0·0000	0·04106	−0·17103	0·12317

TABLE 4. Type 2 solutions with $M_e = 5$, $I_p/I_{0e} = 0{\cdot}5$, $C = 1$, $Pr = 1$

Hypersonic case

For a given stagnation temperature, when $M_e \to \infty$, $I_e \to 0$. The static temperature is approaching zero in the outer part of the boundary layer and the assumed transport property laws are no longer applicable. However in this case it is possible to follow Trella & Libby (1965) who assume that $Pr = 1$ and $C = 1$. The system (5)–(8) is then written in the following form:

$$dv/d\eta = -\tfrac{3}{2}u - \tfrac{3}{2}\hat{\alpha}, \tag{24}$$

$$\frac{d}{d\eta}\left(C\frac{du}{d\eta}\right) = v\frac{du}{d\eta}, \tag{25}$$

$$\frac{d}{d\eta}\left(C\frac{dw^*}{d\eta}\right) = v\frac{dw^*}{d\eta} + w^*(u + \tfrac{3}{2}\hat{\alpha}) - g, \tag{26}$$

$$\frac{d}{d\eta}\left(\frac{C}{Pr}\frac{dg}{d\eta}\right) = v\frac{dg}{d\eta} - 2C\left(\frac{du}{d\eta}\right)^2, \tag{27}$$

where

$$\hat{\alpha} = \tfrac{2}{3}M'T_{0e}/T_e, \tag{28}$$

$$w^* = wT_e/T_{0e}, \tag{29}$$

$$g = TT_e/T_{0e}. \tag{30}$$

The boundary conditions (9) and (10) become:

$$u = v = w^* = 0, \quad g_p = I_p/I_{0e}, \quad \text{when} \quad \eta = 0, \tag{31}$$

$$u \to 1, \quad w^* \to 0, \quad g \to 0, \quad \text{as} \quad \eta \to \infty. \tag{32}$$

The numerical integration of this system also brings out a minimum value $\hat{\alpha}_{cr}$ beyond which there is no solution.

The derivative computed at the wall of both type 1 and 2 solutions is given in tables 5 and 6. Using (28), the existence of a limiting value $\hat{\alpha}_{cr}$ leads to:

$$M'_{cr} = \tfrac{3}{2}\hat{\alpha}_{cr}/[1 + \tfrac{1}{2}(\gamma - 1)M_e^2], \tag{33}$$

which gives the asymptotic behaviour of M'_{cr} when M_e is large. The expression (33) shows that $M'_{cr} \to 0$ when $M_e \to \infty$. Furthermore, the M'_{cr} distribution found for $I_p/I_{0e} = 0{\cdot}5$ and for the Mach number range $7 < M_e < 15$ is shown (figure 4) to match the curve previously obtained in the supersonic case and with variable transport properties very well. For $M_e < 7$ the hypersonic approximation is found to give M'_{cr} correctly down to $M_e = 5$, for which the expression (33) leads to a value differing by less than 8 % from the numerical values found in the supersonic case.

Remarks about the results obtained for $Pr = 1$ (tables 3–6)

By combining the equations (6) and (8) when $Pr = 1$ it is easily found that the variables u and $[T + \tfrac{1}{2}(\gamma - 1)M_e^2 u^2 - T_p]/(T_{0e} - T_p)$ assume the same values for any η, and consequently admit the same derivative at the wall. This property, pointed out by Reshotko (1957) for the case when C = constant, is found to be also valid when $C = C(\eta)$. Therefore the following relation between u'_p and T'_p exists:

$$u'_p = \frac{T'_p}{1 + \tfrac{1}{2}(\gamma - 1)M_e^2}\frac{1}{1 - I_p/I_{0e}}.$$

In the same way, one finds that for the solutions of (16)–(19) $u_p' = g_p'/(1-g_p)$ and with the conditions leading to the results of tables 3 and 4, and 5 and 6, these relations become $u_p' = \frac{1}{3}T_p'$, $u_p' = 2g_p'$ respectively.

$\hat{\alpha}$	u_p'	$w_p^{*\prime}$	g_p'
10·0000	0·91575	0·28777	0·45788
5·0000	0·80852	0·32478	0·40426
1·0000	0·65400	0·39856	0·32700
0·0000	0·57514	0·45078	0·28757
−0·2500	0·54453	0·47495	0·27226
−0·5000	0·50046	0·51467	0·25023
−0·6000	0·47287	0·54312	0·23643
−0·7000	0·40443	0·62966	0·20221
−0·7005	0·39980	0·63652	0·19990

TABLE 5. Type 1 solutions with $M_e \to \infty$, $I_p/I_{0e} = 0{\cdot}5$, $C = 1$, $Pr = 1$

$\hat{\alpha}$	u_p'	$M'w_p^{*\prime}$	g_p'
−0·700	0·39284	−0·45300	0·19642
−0·600	0·30883	−0·48701	0·15441
−0·500	0·26415	−0·47031	0·13208
−0·250	0·16720	−0·36671	0·08360
−0·100	0·10239	−0·25410	0·05119
−0·010	0·04940	−0·13567	0·02470
0·000	0·04106	−0·11402	0·02053

TABLE 6. Type 2 solutions with $M_e \to \infty$, $I_p/I_{0e} = 0{\cdot}5$, $C = 1$, $Pr = 1$

5. Determination of a critical incidence

Detailed tables (Jones 1969) enable us to draw the curves $M' = f(M_e)$ corresponding to different relative incidences i/θ_c for several cone angles (see figure 10). Then the M'_{cr} distribution against M_e shown in figure 4 allows us to obtain, for a given θ_c and for each Mach number value of the external flow, the critical incidence, beyond which there are no solutions in the leeward symmetry plane.

Discussion of the results shown in figure 10

In the case of a large cone angle, $\theta_c > 10°$, M_e tends to a limit when the Mach number at infinity M tends to infinity. These limiting values are approximately 5·75 and 3·50 when $\theta_c = 20°$ and 30° respectively and decreases sharply when θ_c is increasing.

The solutions found for the hypersonic limiting case are only to be used for slender cones ($\theta_c < 10°$). It should also be noticed that in this case the present study shows that solutions exist on the leeward side only for quite small incidences, the critical incidence decreasing as M_e increases.

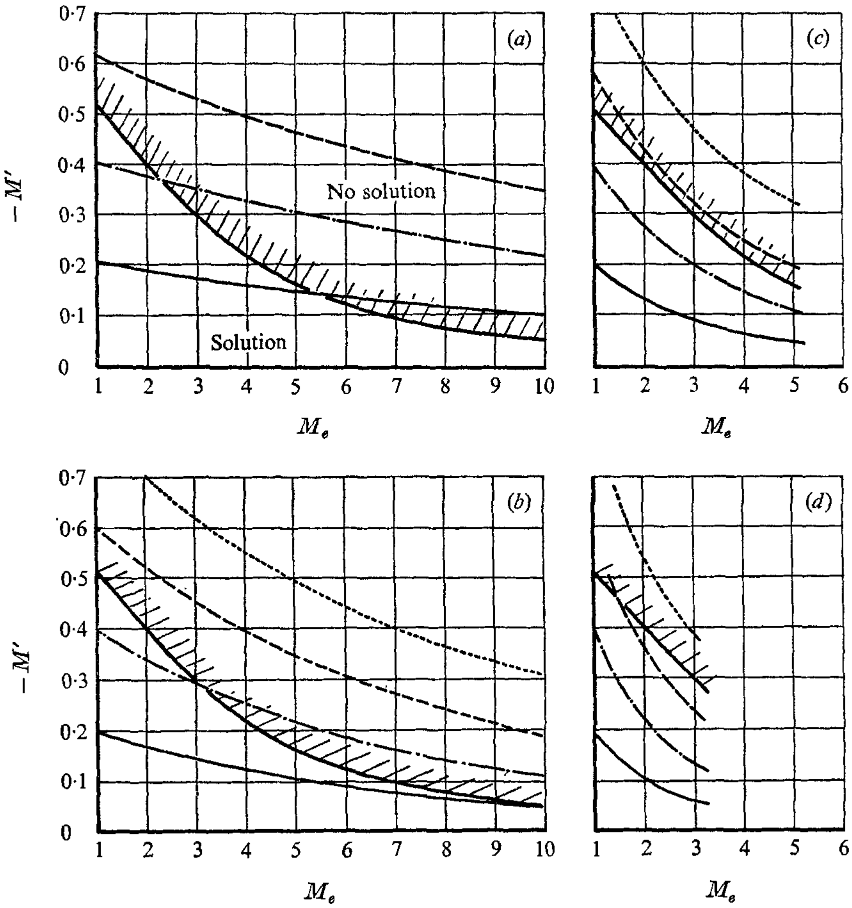

FIGURE 10. Range of existence for solutions on the leeward plane. —, $i/\theta_c = 0{\cdot}1$; —·—, $i/\theta_c = 0{\cdot}2$; – – –, $i/\theta_c = 0{\cdot}3$; - - - - -, $i/\theta_c = 0{\cdot}4$; ////////, $M'_{cr} = f(M_e)$. $\theta_c = 5°, 10°, 20°, 30°$ for (a), (b), (c), (d) respectively.

6. Conclusion

The similar solutions for the boundary-layer equations written in the symmetry plane of a cone at incidence have been investigated in the supersonic case for several Mach numbers of the external flow ($M_e \leqslant 7$). The hypersonic case has also been dealt with, but only when $Pr = 1$ and $C = 1$.

The governing equations in both cases are shown to always have two solutions for each $M' \leqslant 0$ down to a critical value M'_{cr}, below which there is no solution. This limiting value is strongly dependent on M_e. The variation of M'_{cr} is obtained for the whole range $1 \leqslant M_e < \infty$ when $I_p/I_{0e} = 0{\cdot}5$; at the moment we are studying the influence of the parameter I_p/I_{0e}. A knowledge of the curves $M'_{cr} = f(M_e)$ for different I_p/I_{0e} values would allow us to forecast in each case the existence or non-existence of similar solutions near the symmetry plane of a circular cone at incidence as soon as the external flow conditions are known.

The disappearance of the solution on the leeward side at increasing incidence is perhaps connected with the onset of boundary-layer separation observed near the leeward symmetry plane in the case of 9° half-angle cone and for $M_\infty = 7$ (Guffroy, Roux, Marcillat, Brun & Valensi 1968). To determine whether the two phenomena are indeed related to each other it would be necessary to make a systematic experimental study with different values of the cone angle. We are planning to carry out such a study in the case of I_p/I_{0e} equal to 0·5.

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